of the unit square, and is such that $\alpha(C)$ is well defined for any bivariate copula $C$. Examples are measures of monotone association such as Spearman's $\rho_S$ and the correlation of normal scores $\rho_N$. The theory below can be adapted to other types of functionals.

Assume that $a$ is twice continuously differentiable in the region where it is non-zero, and that $a_{12} = \frac{\partial^2 a}{\partial u \partial v}$ is integrable on $(0,1)^2$. With $(u_0, v_0)$ and $(u, v)$ in the interior of the unit square (or on the boundary if $a(\cdot)$ is finite everywhere),

$$a(u,v) - a(u_0,v) - a(u,v_0) + a(u_0,v_0) = \int_{u_0}^{u}\int_{v_0}^{v} a_{12}(s,t)\,\mathrm{d}s\mathrm{d}t.$$

After substituting for $a(u,v)$ and changing the order of integration, (5.15) is the same as:

$$\begin{aligned}\alpha^*(C) = &\int_0^1\int_0^1 a_{12}(s,t)\,C(s,t)\,\mathrm{d}s\mathrm{d}t + \int_{u_0}^1\int_{v_0}^1 a_{12}(s,t)\,\mathrm{d}s\mathrm{d}t\\ &- \int_{u_0}^1\int_0^1 t\,a_{12}(s,t)\,\mathrm{d}s\mathrm{d}t - \int_0^1\int_{v_0}^1 s\,a_{12}(s,t)\,\mathrm{d}s\mathrm{d}t\\ &+ \int_0^1 a(u_0,v)\,\mathrm{d}v + \int_0^1 a(u,v_0)\,\mathrm{d}u - a(u_0,v_0).\end{aligned} \tag{5.16}$$

The sample version of (5.15) is

$$\begin{aligned}\hat\alpha &= \alpha(\widehat{C}_n) = \int_0^1\int_0^1 a(u,v)\,\mathrm{d}\widehat{C}_n(u,v) = n^{-1}\sum_{i=1}^n a\Big(\frac{r_{i1}-\frac12}{n}, \frac{r_{i2}-\frac12}{n}\Big)\\ &= \alpha^*(\widehat{C}_n) + O_p(n^{-1})\\ &= \int_0^1\int_0^1 a(u,v)\,\mathrm{d}\widetilde{C}_n(u,v) + O_p(n^{-1}).\end{aligned}$$

Note that with $S_{in} = [(r_{i1}-1)/n, r_{i1}/n] \times [(r_{i2}-1)/n, r_{i2}/n]$,

$$\int_{(r_{i1}-1)/n}^{r_{i1}/n}\int_{(r_{i2}-1)/n}^{r_{i2}/n} a(u,v)\,\mathrm{d}\widehat{C}_n(u,v) = n^{-1}a\Big(\frac{r_{i1}-\frac12}{n}, \frac{r_{i2}-\frac12}{n}\Big),$$

and from a Taylor expansion to first order,

$$\Big|\int_{S_{in}} a(u,v)\,\widetilde{c}_n(u,v)\,\mathrm{d}u\mathrm{d}v - n^{-1}a\Big(\frac{r_{i1}-\frac12}{n}, \frac{r_{i2}-\frac12}{n}\Big)\Big| \le n^{-2}\max_{(u,v)\in S_{in}}\Big\{\frac{\partial a}{\partial u}, \frac{\partial a}{\partial v}\Big\}.$$

For asymptotics, with the continuous mapping theorem,

$$\begin{aligned}n^{1/2}[\alpha(\widehat{C}_n) - \alpha(C)] &= \int_{[0,1]^2} a_{12}(u,v)\,n^{1/2}\big[\widehat{C}_n(u,v) - C(u,v)\big]\,\mathrm{d}u\mathrm{d}v + O_p(n^{-1/2})\\ &\to_d \int_{[0,1]^2} a_{12}(u,v)\,\mathcal{G}_C(u,v)\,\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Hence $\alpha(\widehat{C}_n)$ is asymptotically normal. If (5.16) doesn't hold, asymptotic normality holds under some conditions and the asymptotic representation has a different form.

An example of a tail-weighted measure of dependence is $\mathrm{Cor}[b(U), b(V)|U > \frac12, V > \frac12]$ if $b$ is increasing on $[\frac12, 1]$ with $b(\frac12) = 0$. For the upper semi-correlation of normal scores, $b(u) = \Phi^{-1}(u)$ for $\frac12 \le u < 1$ (but the second version $\alpha^*$ after integration by parts might not be valid). This conditional correlation involves $\mathbb{E}[b^{m_1}(U)b^{m_2}(V)|U > \frac12, V > \frac12)$. with $(m_1, m_2) \in \{(1,1), (2,0), (0,2), (1,0), (0,1)\}$, that is, (5.15) for five different $a(\cdot)$ functions